

# Noncommutative integrability on noncompact invariant manifolds

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## Abstract

The Mishchenko–Fomenko theorem on noncommutative completely integrable Hamiltonian systems on a symplectic manifold is extended to the case of noncompact invariant submanifolds.

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## 1 Introduction

We are concerned with the classical theorems on abelian and noncommutative integrability of Hamiltonian systems on a symplectic manifold. These are the Liouville–Arnold theorem on abelian completely integrable systems (henceforth CIS) [1, 2, 26], the Poincaré–Lyapounov–Nekhoroshev theorem on abelian partially integrable systems [17, 31, 32], and the Mishchenko–Fomenko one on noncommutative CISs [14, 25, 30]. These theorems state the existence of (generalized) action-angle coordinates around a compact invariant submanifold, which is a torus. However, there is a topological obstruction to the existence of global action-angle coordinates [10, 11]. The Liouville–Arnold and Nekhoroshev theorems have been extended to noncompact invariant submanifolds, which are toroidal cylinders [15, 16, 22, 38]. In particular, this is the case of time-dependent CISs [21, 23]. Any time-dependent CIS of  $m$  degrees of freedom can be represented as an autonomous one of  $m + 1$  degrees of freedom on a homogeneous momentum phase space, where time is a generalized angle coordinate. Therefore, we further consider autonomous CISs.

Our goal here is the following generalization of the Mishchenko–Fomenko theorem to noncommutative CISs whose invariant submanifolds need not be compact.

**Theorem 1.** *Let  $(Z, \Omega)$  be a connected symplectic  $2n$ -dimensional real smooth manifold and  $(C^\infty(Z), \{, \})$  the Poisson algebra of smooth real functions on  $Z$ . Let a subset  $H = (H_1, \dots, H_k)$ ,  $n \leq k < 2n$ , of  $C^\infty(Z)$  obey the following conditions.*

(i) The Hamiltonian vector fields  $\vartheta_i$  of functions  $H_i$  are complete.

(ii) The map  $H : Z \rightarrow \mathbb{R}^k$  is a submersion with connected and mutually diffeomorphic fibers, i.e.,

$$H : Z \rightarrow N = H(Z) \quad (1)$$

is a fibered manifold over a connected open subset  $N \subset \mathbb{R}^k$ .

(iii) There exist real smooth functions  $s_{ij} : N \rightarrow \mathbb{R}$  such that

$$\{H_i, H_j\} = s_{ij} \circ H, \quad i, j = 1, \dots, k. \quad (2)$$

(iv) The matrix function with the entries  $s_{ij}$  (2) is of constant corank  $m = 2n - k$  at all points of  $N$ .

Then the following hold.

(I) The fibers of  $H$  (1) are diffeomorphic to a toroidal cylinder

$$\mathbb{R}^{m-r} \times T^r. \quad (3)$$

(II) Given a fiber  $M$  of  $H$  (1), there exists an open saturated neighbourhood  $U_M$  of it (i.e., a fiber through a point of  $U_M$  belongs to  $U_M$ ), which is a trivial principal bundle with the structure group (3).

(III) Given standard coordinates  $(y^\lambda)$  on the toroidal cylinder (3), the neighbourhood  $U_M$  is provided with bundle coordinates  $(J_\lambda, p_A, q^A, y^\lambda)$ , called the generalized action-angle coordinates, which are the Darboux coordinates of the symplectic form  $\Omega$  on  $U_M$ , i.e.,

$$\Omega = dJ_\lambda \wedge dy^\lambda + dp_A \wedge dq^A. \quad (4)$$

In Hamiltonian mechanics, one can think of functions  $H_i$  in Theorem 1 as being integrals of motion of a CIS. Their level surfaces (fibers of  $H$ ) are invariant submanifolds of a CIS.

## 2 Abelian completely and partially integrable systems

The proof of Theorem 1 is based on the fact that an invariant submanifold of a noncommutative CIS is a maximal integral manifold of some abelian partially integrable Hamiltonian system [14].

If  $k = n$ , Theorem 1 provides the above mentioned extension of the Liouville–Arnold theorem to abelian CISs whose invariant submanifolds are noncompact ([38], Theorem 6.1; [15], Theorem 1). Note that the proof of Theorem 6.1 [38] differs from that of Theorem

1 [15]. It is based on Lemma 6.4. The statement of its Corollary 6.3 is equivalent to the assumption of Lemmas 6.1 – 6.4 that an imbedded invariant submanifold  $N_x \subset K$  admits a Lagrangian transversal submanifold  $W \subset K$  through  $x$ . Apparently, one can avoid the construction of  $T^*(W)$  from the proof and, instead of the map  $\gamma$ , consider the map

$$W \times \mathbb{R}^n \longrightarrow (W \times \mathbb{R}^n) / \mathbb{Z}^{m(x)} \longrightarrow \alpha(W \times \mathbb{R}^n).$$

One also need not appeal to the concept of many-valued functions  $\varphi_i$ , but can show that the fibered manifold  $\alpha(W \times \mathbb{R}^n) \rightarrow W$  is a fiber bundle. This is always true if its fibers are tori and, if  $m(x) < n$ , follows from the fact that sections  $l_i$  of  $W \times \mathbb{R}^n \rightarrow W$  introduced in Lemma 6.2 are smooth.

The condition (ii) of Theorem 1 implies that the functions  $\{H_\lambda\}$  are independent on  $Z$ , i.e., the  $n$ -form  $\overset{n}{\wedge} dH_\lambda$  nowhere vanishes. Accordingly, the Hamiltonian vector fields  $\vartheta_\lambda$  of these functions are independent on  $Z$ , i.e., the multivector field  $\overset{n}{\wedge} \vartheta_\lambda$  nowhere vanishes. If  $k = n$ , these vector fields are mutually commutative, and they span a regular involutive  $n$ -dimensional distribution on  $Z$  whose maximal integral manifolds are exactly fibers of the fibered manifold (1). Thus, every fiber of  $H$  (1) admits  $n$  independent complete vector fields, i.e., it is a locally affine manifold and, consequently, diffeomorphic to a toroidal cylinder.

Considering an abelian CIS around some compact invariant submanifold, we come to the Liouville–Arnold theorem (somebody also calls it the Liouville–Mineur–Arnold theorem [39]). Instead of the conditions (i) and (ii) of Theorem 1, one can suppose that integrals of motion  $\{H_\lambda\}$  are independent almost everywhere on a symplectic manifold  $Z$ , i.e., the set of points where the exterior form  $\overset{n}{\wedge} dH_\lambda$  (or, equivalently, the multivector field  $\overset{n}{\wedge} \vartheta_\lambda$ ) vanishes is nowhere dense. In this case, connected components of level surfaces of functions  $\{H_\lambda\}$  form a singular Stefan foliation  $\mathcal{F}$  of  $Z$  whose leaves are both the maximal integral manifolds of the singular involutive distribution spanned by the vector fields  $\vartheta_\lambda$  and the orbits of the pseudogroup  $G$  of local diffeomorphisms of  $Z$  generated by flows of these vector fields [36, 37]. Let  $M$  be a leaf of  $\mathcal{F}$  through a regular point  $z \in Z$  where  $\overset{n}{\wedge} \vartheta_\lambda \neq 0$ . It is regular everywhere because the group  $G$  preserves  $\overset{n}{\wedge} \vartheta_\lambda$ . If  $M$  is compact and connected, there exists its saturated open neighbourhood  $U_M$  such that the map  $H$  restricted to  $U_M$  satisfies the condition (ii) of Theorem 1, i.e., the foliation  $\mathcal{F}$  of  $U_M$  is a fibered manifold in tori  $T^n$ . Since its fibers are compact,  $U_M$  is a bundle [29]. Hence, it contains a saturated open neighbourhood of  $M$ , say again  $U_M$ , which is a trivial principal bundle with the structure group  $T^n$ . Providing  $U_M$  with the Darboux (action-angle) coordinates  $(J_\lambda, \alpha^\lambda)$ , one uses the fact that there are no linear functions on a torus  $T^n$ .

The Poincaré–Lyapounov–Nekhoroshev theorem generalizes the Liouville–Arnold one to partially integrable systems characterized by  $k < n$  independent integrals of motion  $H_\lambda$  in involution. In this case, one deals with  $k$ -dimensional maximal integral manifolds of the distribution spanned by Hamiltonian vector fields  $\vartheta_\lambda$  of integrals of motion  $H_\lambda$ . The Poincaré–Lyapounov–Nekhoroshev theorem imposes a sufficient condition which Hamiltonian vector fields  $\vartheta_\lambda$  must satisfy in order that their compact maximal integral manifold  $M$  admits an open neighbourhood fibered in tori [17, 18]. Such a condition has been also investigated in the case of noncommutative vector fields depending on parameters [19]. Extending the Poincaré–Lyapounov–Nekhoroshev theorem to the case of noncompact integral submanifolds, we in fact assumed from the beginning that these submanifolds form a fibration [15, 22, 23]. In a more general setting, we have studied the property of a given dynamical system to be Hamiltonian relative to different Poisson structures [5, 22, 35]. As is well known, any integrable Hamiltonian system is Hamiltonian relative to different symplectic and Poisson structures, whose variety has been analyzed from different viewpoints [3, 4, 8, 12, 13, 28, 34]. One of the reasons is that bi-Hamiltonian systems have a large supply of integrals of motion. Here, we refer to our following result on partially integrable systems on a symplectic manifold ([22], Theorem 6).

**Theorem 2.** *Given a  $2n$ -dimensional symplectic manifold  $(Z, \Omega)$ , let  $\{H_1, \dots, H_m\}$ ,  $m \leq n$ , be smooth real functions on  $Z$  in involution which satisfy the following conditions.*

- (i) The functions  $H_\lambda$  are everywhere independent.*
- (ii) Their Hamiltonian vector fields  $\vartheta_\lambda$  are complete.*
- (iii) These vector fields span a regular distribution whose maximal integral manifolds form a fibration  $\mathcal{F}$  of  $Z$  with diffeomorphic fibers.*

*Then the following hold.*

- (I) All fibers of  $\mathcal{F}$  are diffeomorphic to a toroidal cylinder (3).*
- (II) There is an open saturated neighbourhood  $U_M$  of every fiber  $M$  of  $\mathcal{F}$  which is a trivial principal bundle with the structure group (3).*
- (III) Given standard coordinates  $(y^\lambda)$  on the toroidal cylinder (3), the neighbourhood  $U_M$  is endowed with the bundle coordinates  $(J_\lambda, p_A, q^A, y^\lambda)$  such that the symplectic form  $\Omega$  is brought into the form (4).*

Theorem 2 provides the above mentioned generalization of the Poincaré–Lyapounov–Nekhoroshev theorem to the case of noncompact invariant submanifolds. A geometric aspect of this generalization is the following. Any fibered manifold whose fibers are diffeomorphic either to  $\mathbb{R}^r$  or a compact connected manifold  $K$  (e.g., a torus) is a fiber bundle

[29]. However, a fibered manifold whose fibers are diffeomorphic to a product  $\mathbb{R}^r \times K$  (e.g., a toroidal cylinder (3)) need not be a fiber bundle (see [20], Example 1.2.2).

### 3 The proof of Theorem 1

Theorem 2 is the final step of the proof of Theorem 1. The condition (iv) of Theorem 1 implies that an  $m$ -dimensional invariant submanifold of a noncommutative CIS is a maximal integral manifold of some abelian partially integrable Hamiltonian system obeying the conditions of Theorem 2. The proof of this fact is based on the following two assertions [14, 27].

**Lemma 3.** *Given a symplectic manifold  $(Z, \Omega)$ , let  $H : Z \rightarrow N$  be a fibered manifold such that, for any two functions  $f, f'$  constant on fibers of  $H$ , their Poisson bracket  $\{f, f'\}$  is so. Then  $N$  is provided with an unique coinduced Poisson structure  $\{, \}_N$  such that  $H$  is a Poisson morphism.*

Since any function constant on fibers of  $H$  is a pull-back of some function on  $N$ , the condition of Lemma 3 is satisfied due to item (iii) of Theorem 1. Thus, the base  $N$  of the fibration (1) is endowed with a coinduced Poisson structure.

**Lemma 4.** *Given a fibered manifold  $H : Z \rightarrow N$ , the following conditions are equivalent:*  
*(i) the rank of the coinduced Poisson structure  $\{, \}_N$  on  $N$  equals  $2\dim N - \dim Z$ ,*  
*(ii) the fibers of  $H$  are isotropic,*  
*(iii) the fibers of  $H$  are maximal integral manifolds of the involutive distribution spanned by the Hamiltonian vector fields of the pull-back  $H^*C$  of Casimir functions  $C$  of the Poisson algebra on  $N$ .*

It is readily observed that the condition (i) of Lemma 4 is satisfied due to the assumption (iv) of Theorem 1. It follows that every fiber  $M$  of the fibration (1) is a maximal integral manifold of the involutive distribution spanned by the Hamiltonian vector fields  $v_\lambda$  of the pull-back  $H^*C_\lambda$  of  $m$  independent Casimir functions  $\{C_1, \dots, C_m\}$  on an open neighbourhood  $N_M$  of the point  $H(M)$ . Let us put  $U_M = H^{-1}(N_M)$ . It is an open saturated neighbourhood of  $M$ . Since

$$H^*C_\lambda(z) = (C_\lambda \circ H)(z) = C_\lambda(H_i(z)), \quad z \in U_M, \quad (5)$$

the Hamiltonian vector fields  $v_\lambda$  on  $M$  are linear combinations of Hamiltonian vector fields  $\vartheta_i$  of the functions  $H_i$  and, therefore, they are complete on  $M$ . Similarly, they are complete

on any fiber of  $U_M$  and, consequently, on  $U_M$ . Thus, the conditions of Theorem 2 hold on  $U_M$ . This completes the proof of Theorem 1.

The proof of Theorem 1 gives something more. Let  $\{H_i\}$  be integrals of motion of a Hamiltonian  $\mathcal{H}$ . Since  $(J_\lambda, p_A, q^A)$  are coordinates on  $N$ , they are also integrals of motion of  $\mathcal{H}$ . Therefore, the Hamiltonian  $\mathcal{H}$  depends only on the action coordinates  $J_\lambda$ , and the equation of motion read

$$\dot{y}^\lambda = \frac{\partial \mathcal{H}}{\partial J_\lambda}, \quad J_\lambda = \text{const.}, \quad q^A = \text{const.}, \quad p_A = \text{const.}$$

Though the integrals of motion  $H_i$  are smooth functions of coordinates  $(J_\lambda, q^A, p_A)$ , the Casimir functions

$$C_\lambda(H_i(J_\mu, q^A, p_A)) = C_\lambda(J_\mu)$$

depend only on the action coordinates  $J_\lambda$ . Moreover, a Hamiltonian

$$\mathcal{H}(J_\mu) = \mathcal{H}(C_\lambda(H_i(J_\mu, q^A, p_A)))$$

is expressed in integrals of motion  $H_i$  through the Casimir functions (5).

Let us note that, under the assumptions of the Mishchenko–Fomenko theorem, a non-commutative CIS is also integrable in the abelian sense. Namely, it admits  $n$  independent integrals of motion in involution [6]. Under the conditions of Theorem 1, such integrals of motion in involution exist, too. All of them are the pull-back of functions on  $N$ . However, one must justify that they obey the condition (iii) of Theorem 2 in order to characterize an abelian CIS.

## 4 Example

The original Mishchenko–Fomenko theorem is restricted to CISs whose integrals of motion  $\{H_1, \dots, H_k\}$  form a  $k$ -dimensional real Lie algebra  $\mathcal{G}$  of rank  $m$  with the commutation relations

$$\{H_i, H_j\} = c_{ij}^h H_h, \quad c_{ij}^h = \text{const.}$$

In this case, nonvanishing complete Hamiltonian vector fields  $\vartheta_i$  of  $H_i$  define a free Hamiltonian action on  $Z$  of some connected Lie group  $G$  whose Lie algebra is isomorphic to  $\mathcal{G}$ . Orbits of  $G$  coincide with  $k$ -dimensional maximal integral manifolds of the regular distribution on  $Z$  spanned by Hamiltonian vector fields  $\vartheta_i$  [37]. Furthermore, one can treat  $H$

(1) as an equivariant momentum mapping of  $Z$  to the Lie coalgebra  $\mathcal{G}^*$ , provided with the coordinates  $x_i(H(z)) = H_i(z)$ ,  $z \in Z$ , [23, 24]. In this case, the coinduced Poisson structure  $\{\cdot, \cdot\}_N$  in Lemma 3 coincides with the canonical Lie–Poisson structure on  $\mathcal{G}^*$  given by the Poisson bivector field

$$w = \frac{1}{2} c_{ij}^h x_h \partial^i \wedge \partial^j.$$

Recall that the coadjoint action of  $\mathcal{G}$  on  $\mathcal{G}^*$  reads  $\varepsilon_i(x_j) = c_{ij}^h x_h$ , where  $\{\varepsilon_i\}$  is a basis for  $\mathcal{G}$ . Casimir functions of the Lie–Poisson structure are exactly the coadjoint invariant functions on  $\mathcal{G}^*$ . They are constant on orbits of the coadjoint action of  $G$  on  $\mathcal{G}^*$  which coincide with leaves of the symplectic foliation of  $\mathcal{G}^*$ . Given a point  $z \in Z$  and the orbit  $G_z$  of  $G$  in  $Z$  through  $z$ , the fibration  $H$  (1) projects this orbit onto the orbit  $G_{H(z)}$  of the coadjoint action of  $G$  in  $\mathcal{G}^*$  through  $H(z)$ . Moreover, by virtue of item (iii), Lemma 4, the inverse image  $H^{-1}(G_{H(z)})$  of  $G_{H(z)}$  coincides with the orbit  $G_z$ . It follows that any orbit of  $G$  in  $Z$  is fibered in invariant submanifolds.

The Mishchenko–Fomenko theorem has been mainly applied to CISs whose integrals of motion form a compact Lie algebra. Indeed, the group  $G$  generated by flows of the Hamiltonian vector fields is compact, and every orbit of  $G$  in  $Z$  is compact. Since a fibration of a compact manifold possesses compact fibers, any invariant submanifold of such a noncommutative CIS is compact. Therefore, our Theorem 1 essentially extends a class of noncommutative CISs under investigation.

For instance, a spherical top exemplifies a noncommutative CIS whose integrals of motion make up the compact Lie algebra  $so(3)$  with respect to some symplectic structure.

Let us consider a CIS with the Lie algebra  $\mathcal{G} = so(2, 1)$  of integrals of motion  $\{H_1, H_2, H_3\}$  on a four-dimensional symplectic manifold  $(Z, \Omega)$ , namely,

$$\{H_1, H_2\} = -H_3, \quad \{H_2, H_3\} = H_1, \quad \{H_3, H_1\} = H_2. \quad (6)$$

The rank of this algebra (the dimension of its Cartan subalgebra) equals one. Therefore, an invariant submanifold in Theorem 1 is  $M = \mathbb{R}$ , provided with a Cartesian coordinate  $y$ . Let us consider its open saturated neighbourhood  $U_M$  projected via  $H : U_M \rightarrow N$  onto a domain  $N \subset \mathcal{G}^*$  in the Lie coalgebra  $\mathcal{G}^*$  centered at a point  $H(M) \in \mathcal{G}^*$  which belongs to an orbit of the coadjoint action of maximal dimension 2. A domain  $N$  is endowed with the coordinates  $(x_1, x_2, x_3)$  such that integrals of motion  $\{H_1, H_2, H_3\}$  on  $U_M = N \times \mathbb{R}$ , coordinated by  $(x_1, x_2, x_3, y)$ , read

$$H_1 = x_1, \quad H_2 = x_2, \quad H_3 = x_3.$$

As was mentioned above, the coinduced Poisson structure on  $N$  is the Lie–Poisson structure

$$w = x_2 \partial^3 \wedge \partial^1 - x_3 \partial^1 \wedge \partial^2 + x_1 \partial^2 \wedge \partial^3. \quad (7)$$

Let us endow  $N$  with different coordinates  $(r, x_1, \gamma)$  given by the equalities

$$r = (x_1^2 + x_2^2 - x_3^2)^{1/2}, \quad x_2 = (r^2 - x_1^2)^{1/2} \text{ch} \gamma, \quad x_3 = (r^2 - x_1^2)^{1/2} \text{sh} \gamma, \quad (8)$$

where  $r$  is a Casimir function on  $\mathcal{G}^*$ . It is readily observed that the coordinates (8) are the Darboux coordinates of the Lie–Poisson structure (7), namely,

$$w = \frac{\partial}{\partial \gamma} \wedge \frac{\partial}{\partial x_1}. \quad (9)$$

Let  $\vartheta_r$  be the Hamiltonian vector field of the Casimir function  $r$  (8). This vector field is a combination

$$\vartheta_r = \frac{1}{r} (x_1 \vartheta_1 + x_2 \vartheta_2 - x_3 \vartheta_3)$$

of the Hamiltonian vector fields  $\vartheta_i$  of integrals of motion  $H_i$ . Its flows are invariant submanifolds. Let  $y$  be a parameter along the flows of this vector field, i.e.,

$$\vartheta_r = \frac{\partial}{\partial y}.$$

Then the Poisson bivector associated to the symplectic form  $\Omega$  on  $U_M$  is

$$W = \frac{\partial}{\partial r} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial \gamma} \wedge \frac{\partial}{\partial x_1}. \quad (10)$$

Accordingly, Hamiltonian vector fields of integrals of motion take the form

$$\begin{aligned} \vartheta_1 &= -\frac{\partial}{\partial \gamma}, \\ \vartheta_2 &= r(r^2 - x_1^2)^{-1/2} \text{ch} \gamma \frac{\partial}{\partial y} + x_1(r^2 - x_1^2)^{-1/2} \text{ch} \gamma \frac{\partial}{\partial \gamma} + (r^2 - x_1^2)^{1/2} \text{sh} \gamma \frac{\partial}{\partial x_1}, \\ \vartheta_3 &= r(r^2 - x_1^2)^{-1/2} \text{sh} \gamma \frac{\partial}{\partial y} + x_1(r^2 - x_1^2)^{-1/2} \text{sh} \gamma \frac{\partial}{\partial \gamma} + (r^2 - x_1^2)^{1/2} \text{ch} \gamma \frac{\partial}{\partial x_1}. \end{aligned}$$

Thus, a symplectic annulus  $(U_M, W)$  around an invariant submanifold  $M = \mathbb{R}$  is endowed with the generalized action-angle coordinates  $(r, x_1, \gamma, y)$ , and possesses the corresponding noncommutative CIS  $\{r, H_1, \gamma\}$  with the commutation relations

$$\{r, H_1\} = 0, \quad \{r, \gamma\} = 0, \quad \{H_1, \gamma\} = 1.$$

This CIS is related to the original one by the transformations

$$r = (H_1^2 + H_2^2 - H_3^2)^{1/2}, \quad H_2 = (r^2 - H_1^2)^{1/2} \text{ch} \gamma, \quad H_3 = (r^2 - H_1^2)^{1/2} \text{sh} \gamma.$$

Its Hamiltonian is expressed only in the action variable  $r$ .



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